European Journal of Combinatorics 32 (2011) 226–232

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Ehrhart polynomials of convex polytopes with small volumes

Takayuki Hibi, Akihiro Higashitani, Yuuki Nagazawa

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

ARTICLE INFO

Article history:
Received 15 August 2009
Accepted 30 September 2010
Available online 27 October 2010

ABSTRACT

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ be its δ-vector. By using the known inequalities on δ-vectors, we classify the possible δ-vectors of convex polytopes of dimension d with $\sum_{i=0}^{d} \delta_i \leq 3$.

0. Introduction

One of the most attractive problems on enumerative combinatorics of convex polytopes is to find a combinatorial characterizations of the Ehrhart polynomials of integral convex polytopes. First of all, we recall what the Ehrhart polynomial of a convex polytope is.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* convex polytope; i.e., a convex polytope any of whose vertices has integer coordinates, of dimension d, and let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer n we define the numerical functions $i(\mathcal{P}, n)$ and $i^*(\mathcal{P}, n)$ by setting

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|.$$

Here $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X.

The systematic study of $i(\mathcal{P}, n)$ originated in the work of Ehrhart [1], who established the following fundamental properties:

(0.1) $i(\mathcal{P}, n)$ is a polynomial in n of degree d (and thus in particular $i(\mathcal{P}, n)$ can be defined for *every* integer n);
(0.2) $i(\mathcal{P}, 0) = 1$;
(0.3) *(loi de réciprocité)* $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of $\mathcal{P}$. An introduction to the theory of Ehrhart polynomials is discussed in [6, pp. 235–241] and [2, Part II].

E-mail addresses: hibi@math.sci.osaka-u.ac.jp (T. Hibi), sm5037ha@ecs.cmc.osaka-u.ac.jp (A. Higashitani), sm5032ny@ecs.cmc.osaka-u.ac.jp (Y. Nagazawa).

0195-6698/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2010.10.003

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1+\sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i. \tag{1}$$

Then the basic facts (0.1) and (0.2) on $i(\mathcal{P}, n)$ together with a fundamental result on the generating function [6, Corollary 4.3.1] guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

which appears in Eq. (1) is the *δ-vector* of $\mathcal{P}$. Thus $\delta_0 = 1$ and $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$.

It follows from *loi de réciprocité* (0.3) that

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n)\lambda^n = \frac{\sum_{i=0}^{d} \delta_{d-i}\lambda^{i+1}}{(1-\lambda)^{d+1}}. \tag{2}$$

In particular $\delta_d = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Hence $\delta_1 \geq \delta_d$. Moreover, each δ_i is nonnegative [7]. In addition, if $(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N$ is nonempty, i.e., $\delta_d \neq 0$, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i < d$ [3]. When $d = N$, the leading coefficient $(\sum_{i=0}^{d} \delta_i)/d!$ of $i(\mathcal{P}, n)$ is equal to the usual volume $\mathrm{vol}(\mathcal{P})$ of $\mathcal{P}$ [6, Proposition 4.6.30].

It follows from Eq. (2) that

$$\max\{i : \delta_i \neq 0\} + \min\{i : i(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N \neq \emptyset\} = d+1.$$

Let $s = \max\{i : \delta_i \neq 0\}$. Stanley [8] shows the inequalities

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq [s/2] \tag{3}$$

by using the theory of Cohen–Macaulay rings. On the other hand, the inequalities

$$\delta_{d-1} + \delta_{d-2} + \cdots + \delta_{d-i} \leq \delta_2 + \delta_3 + \cdots + \delta_i + \delta_{i+1}, \quad 1 \leq i \leq [(d-1)/2] \tag{4}$$

appear in [3, Remark (1.4)]. These inequalities (3) and (4) are discussed in detail by Stapledon [5].

Somewhat surprisingly, when $\sum_{i=0}^{d} \delta_i \leq 3$, the above inequalities (3) together with (4) give a characterization of the possible δ-vectors. In fact,

Theorem 0.1. *Let $d \geq 3$. Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$ and $\delta_1 \geq \delta_d$, which satisfies $\sum_{i=0}^{d} \delta_i \leq 3$, there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $(\delta_0, \delta_1, \ldots, \delta_d)$ satisfies all inequalities (3) and (4).*

The "Only if" part of Theorem 0.1 is obvious. In addition, no discussion will be required for the case of $\sum_{i=0}^{d} \delta_i = 1$. The "If" part of Theorem 0.1 will be given in Section 2 for the case of $\sum_{i=0}^{d} \delta_i = 2$ and in Section 3 for the case of $\sum_{i=0}^{d} \delta_i = 3$.

On the other hand, Example 1.2 shows that Theorem 0.1 is no longer true for the case of $\sum_{i=0}^{d} \delta_i = 4$. Finally, when $d \leq 2$, the possible δ-vectors are known [4].

1. Review on the computation of the δ-vector of a simplex

We recall from [2, Part II] the well-known combinatorial technique for computing the δ-vector of a simplex.

- Given an integral d-simplex $\mathcal{F} \subset \mathbb{R}^N$ with the vertices $v_0, v_1, \ldots, v_d$, we set $\widetilde{\mathcal{F}} = \{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{F}\}$, which is an integral d-simplex in $\mathbb{R}^{N+1}$ with the vertices $(v_0, 1), (v_1, 1), \ldots, (v_d, 1)$ and $\partial\widetilde{\mathcal{F}} = \{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \partial\mathcal{F}\}$ is its boundary. Clearly $i(\mathcal{F}, n) = i(\widetilde{\mathcal{F}}, n)$ and $i^*(\mathcal{F}, n) = i^*(\widetilde{\mathcal{F}}, n)$ for all n.
- The subset $\mathcal{C} = \mathcal{C}(\widetilde{\mathcal{F}}) \subset \mathbb{R}^{N+1}$ defined by $\mathcal{C} = \{r\beta : \beta \in \widetilde{\mathcal{F}}, 0 \leq r \in \mathbb{Q}\}$ is called *the simplicial cone associated with $\mathcal{F} \subset \mathbb{R}^N$ with apex* $(0, \ldots, 0)$. Its boundary is $\partial\mathcal{C} = \{r\beta : \beta \in \partial\widetilde{\mathcal{F}}, 0 \leq r \in \mathbb{Q}\}$. One has $i(\mathcal{F}, n) = |\{(\alpha, n) \in \mathcal{C} : \alpha \in \mathbb{Z}^N\}|$ and $i^*(\mathcal{F}, n) = |\{(\alpha, n) \in \mathcal{C} \setminus \partial\mathcal{C} : \alpha \in \mathbb{Z}^N\}|$.

- Each rational point $\alpha \in \mathcal{C}$ has a unique expression of the form $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$ with each $0 \le r_i \in \mathbb{Q}$. Moreover, each rational point $\alpha \in \mathcal{C}\backslash\partial\mathcal{C}$ has a unique expression of the form $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$ with each $0 < r_i \in \mathbb{Q}$.
- Let S (resp. S^*) be the set of all points $\alpha \in \mathcal{C} \cap \mathbb{Z}^{N+1}$ (resp. $\alpha \in (\mathcal{C} \setminus \partial\mathcal{C}) \cap \mathbb{Z}^{N+1}$) of the form $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$, where each $r_i \in \mathbb{Q}$ with $0 \le r_i < 1$ (resp. with $0 < r_i \le 1$).
- The degree of an integer point $(\alpha, n) \in \mathcal{C}$ is $\deg(\alpha, n) = n$.

Lemma 1.1. (a) *Let δ_i be the number of integer points $\alpha \in S$ with* $\deg \alpha = i$. *Then*

$$1 + \sum_{n=1}^{\infty} i(\mathcal{F}, n)\lambda^n = \frac{\delta_0 + \cdots + \delta_d \lambda^d}{(1-\lambda)^{d+1}}.$$

(b) *Let δ_i^* be the number of integer points $\alpha \in S^*$ with* $\deg \alpha = i$. *Then*

$$\sum_{n=1}^{\infty} i^*(\mathcal{F}, n)\lambda^n = \frac{\delta_1^* \lambda + \cdots + \delta_{d+1}^* \lambda^{d+1}}{(1-\lambda)^{d+1}}.$$

(c) *One has $\delta_i^* = \delta_{(d+1)-i}$ for each $1 \le i \le d+1$.*

Example 1.2. Theorem 0.1 is no longer true for the case of $\sum_{i=0}^{d} \delta_i = 4$. In fact, the sequence $(1, 0, 1, 0, 1, 1, 0, 0)$ cannot be the δ-vector of an integral convex polytope of dimension 7. Suppose, on the contrary, that there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ with $(\delta_0, \delta_1, \ldots, \delta_7) = (1, 0, 1, 0, 1, 1, 0, 0)$ its δ-vector. Since $\delta_1 = 0$, we know that $\mathcal{P}$ is a simplex. Let $v_0, v_1, \ldots, v_7$ be the vertices of $\mathcal{P}$. By using Lemma 1.1, one has $S = \{(0, \ldots, 0), (\alpha, 2), (\beta, 4), (\gamma, 5)\}$ and $S^* = \{(\alpha', 3), (\beta', 4), (\gamma', 6), (\sum_{i=0}^{7} v_i, 8)\}$. Write $\alpha' = \sum_{i=0}^{7} r_i v_i$ with each $0 < r_i \le 1$. Since $(\alpha', 3) \notin S$, there is $0 \le j \le 7$ with $r_j = 1$. If there are $0 \le k < \ell \le 7$ with $r_k = r_\ell = 1$, say, $r_0 = r_1 = 1$, then $0 < r_q < 1$ for each $2 \le q \le 7$ and $\sum_{i=2}^{7} r_i = 1$. Hence $(\alpha' - v_0 - v_1, 1) \in S$, a contradiction. Thus there is a unique $0 \le j \le 7$ with $r_j = 1$, say, $r_0 = 1$. Then $\alpha = \sum_{i=1}^{7} r_i v_i$ and $\gamma = \sum_{i=1}^{7}(1 - r_i)v_i$. Let $\mathcal{F}$ denote the facet of $\mathcal{P}$ whose vertices are $v_1, v_2, \ldots, v_7$ with $\delta(\mathcal{F}) = (\delta'_0, \delta'_1, \ldots, \delta'_6) \in \mathbb{Z}^7$. Then $\delta'_2 = \delta'_5 = 1$. Since $\delta'_i \le \delta_i$ for each $0 \le i \le 6$, it follows that $\delta(\mathcal{F}) = (1, 0, 1, 0, 0, 1, 0)$. This contradicts the inequalities (3).

2. A proof of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 2$

The goal of this section is to prove the "If" part of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 2$. First of all, we recall the following well-known lemma:

Lemma 2.1. *Suppose that $(\delta_0, \delta_1, \ldots, \delta_d)$ is the δ-vector of an integral convex polytope of dimension d. Then there exists an integral convex polytope of dimension $d+1$ whose δ-vector is $(\delta_0, \delta_1, \ldots, \delta_d, 0)$.*

Proof. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\mathcal{Q} \subset \mathbb{R}^{N+1}$ the convex hull of $\{(\alpha, 0) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\}$ together with $(0, \ldots, 0, 1) \in \mathbb{R}^{N+1}$. Then $\mathcal{Q}$ is an integral convex polytope of dimension $d+1$. It follows that

$$i(\mathcal{Q}, n) = \sum_{q=0}^{n} i(\mathcal{P}, q), \quad n = 0, 1, 2, \ldots,$$

where $i(\mathcal{P}, 0) = i(\mathcal{Q}, 0) = 1$. Hence

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} i(\mathcal{Q}, n)\lambda^n &= \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right](1 + \lambda + \lambda^2 + \cdots) \\ &= \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right](1-\lambda)^{-1}. \end{aligned}$$

Thus

$$(1-\lambda)^{d+2}\left[1+\sum_{n=1}^{\infty} i(\mathcal{Q},n)\lambda^n\right] = (1-\lambda)^{d+1}\left[1+\sum_{n=1}^{\infty} i(\mathcal{P},n)\lambda^n\right],$$

as desired. □

Let $d \geq 3$. We study a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers with $\delta_0 = 1$ and $\delta_1 \geq \delta_d$ which satisfies all inequalities (4) together with $\sum_{i=0}^{d} \delta_i = 2$. Since $\delta_0 = 1$, $\delta_1 \geq \delta_d$ and $\sum_{i=0}^{d} \delta_i = 2$, one has $\delta_d = 0$. Hence there is an integer $i \in \{1, \ldots, [(d+1)/2]\}$ such that $(\delta_0, \delta_1, \ldots, \delta_d) = (1, 0, \ldots, 0, \underbrace{1}_{i\text{-th}}, 0, \ldots, 0)$, where $\underbrace{1}_{i\text{-th}}$ stands for $\delta_i = 1$. By virtue of Lemma 2.1 our work is to find an integral convex polytope $\mathcal{P}$ of dimension d with $(1, 0, \ldots, 0, \underbrace{1}_{((d+1)/2)\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ its δ-vector.

Let $\mathcal{P} \subset \mathbb{R}^d$ be the integral simplex of dimension d whose vertices $v_0, v_1, \ldots, v_d$ are

$$v_i = \begin{cases} (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, 0, \ldots, 0), & i = 1, \ldots, d-1, \\ (1, 0, \ldots, 0, 1), & i = d, \\ (0, 0, \ldots, 0), & i = 0. \end{cases}$$

When d is odd, one has $\mathrm{vol}(\mathcal{P}) = 2/d!$ by using an elementary linear algebra. Since

$$\frac{1}{2}\{(v_0, 1) + (v_1, 1) + \cdots + (v_d, 1)\} = (1, 1, \ldots, 1, (d+1)/2) \in \mathbb{Z}^{d+1},$$

Lemma 1.1 says that $\delta_{(d+1)/2} \geq 1$. Thus, since $\mathrm{vol}(\mathcal{P}) = 2/d!$, one has

$$\delta(\mathcal{P}) = (1, 0, \ldots, 0, \underbrace{1}_{((d+1)/2)\text{-th}}, 0, \ldots, 0),$$

as desired.

3. A proof of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 3$

The goal of this section is to prove the "If" part of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 3$. Let $d \geq 3$. Suppose that a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers with $\delta_0 = 1$ and $\delta_1 \geq \delta_d$ satisfies all inequalities (3) and (4) together with $\sum_{i=0}^{d} \delta_i = 3$.

When there is $1 \leq i \leq d$ with $\delta_i = 2$, the same discussion as in Section 1 can be applied. In fact, instead of the vertices of the convex polytope arising in the last paragraph of Section 1, we may consider the convex polytope whose vertices $v_0, v_1, \ldots, v_d$ are

$$v_i = \begin{cases} (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, 0, \ldots, 0), & i = 1, \ldots, d-1, \\ (2, 0, \ldots, 0, 1), & i = d, \\ (0, 0, \ldots, 0), & i = 0. \end{cases}$$

Now, in what follows, a sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ with each $\delta_i \in \{0, 1\}$, where $\delta_0 = 1$ and $\delta_1 \geq \delta_d$, which satisfies all inequalities (3) and (4) together with $\sum_{i=0}^{d} \delta_i = 3$ will be considered.

If $\delta_d = 1$, then $\delta_1 = 1$. However, since $d \geq 3$, this contradicts (3). If $\delta_1 = 1$, then $\delta_2 = 1$ by (3). Clearly, $(1, 1, 1, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ is a possible δ-vector. Thus we will assume that $\delta_1 = \delta_d = 0$. Let $\delta_m = \delta_n = 1$ with $1 < m < n < d$. Let $p = m - 1$, $q = n - m - 1$, and $r = d - n$. By (3) one has $0 \leq q \leq p$. Moreover, by (4) one has $p \leq r$. Consequently,

$$0 \leq q \leq p \leq r, \quad p + q + r = d - 2. \tag{5}$$

Our work is to construct an integral convex polytope $\mathcal{P}$ with dimension d whose δ-vector coincides with $\delta(\mathcal{P}) = (1, \underbrace{0, \ldots, 0}_{p}, 1, \underbrace{0, \ldots, 0}_{q}, 1, \underbrace{0, \ldots, 0}_{r})$ for an arbitrary integer $1 < m < n < d$ satisfying the conditions (5).

Lemma 3.1. *Let $d = 3k + 2$. There exists an integral convex polytope $\mathcal{P}$ of dimension d whose δ-vector coincides with*

$$(1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k}) \in \mathbb{Z}^{d+1}.$$

Proof. When $k \geq 1$, let $\mathcal{P} \subset \mathbb{R}^d$ be the integral simplex of dimension d with the vertices $v_0, v_1, \ldots, v_d$, where

$$v_i = \begin{cases} (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, \underbrace{1}_{(i+2)\text{-th}}, 0, \ldots, 0), & i = 1, \ldots, d-2, \\ (1, 0, \ldots, 0, 1, 1), & i = d-1, \\ (1, 1, 0, \ldots, 0, 1), & i = d, \\ (0, \ldots, 0), & i = 0. \end{cases}$$

By using the induction on k it follows that $\mathrm{vol}(\mathcal{P}) = 3/d!$. Since

$$\frac{1}{3}\{(v_0, 1) + (v_1, 1) + \cdots + (v_d, 1)\} = (1, 1, \ldots, 1, k+1) \in \mathbb{Z}^{d+1},$$

Lemma 1.1 now guarantees that $\delta_{k+1} \geq 1$ and $\delta^*_{k+1} \geq 1$. Hence $\delta_{k+1} = 1$ and $\delta_{2k+2} = 1$, as required. $\square$

Lemma 3.2. *Let $d = 3k + 2$, $\ell > 0$ and $d' = d + 2\ell$. There exists an integral simplex $\mathcal{P} \subset \mathbb{R}^{d'}$ of dimension d' whose δ-vector coincides with*

$$(1, \underbrace{0, \ldots, 0}_{k+\ell}, 1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k+\ell}) \in \mathbb{Z}^{d'+1}.$$

Proof. *(First Step)* Let $k = 0$. Thus $d = 2$ and $d' = 2\ell + 2$. Let $\mathcal{P} \subset \mathbb{R}^{d'}$ be an integer convex polytope of dimension d' whose vertices $v_0, v_1, \ldots, v_{2\ell+2}$ are

$$v_i = \begin{cases} (2, 1, 0, 0, \ldots, 0), & i = 1, \\ (0, 2, 1, 0, \ldots, 0), & i = 2, \\ (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, 0, \ldots, 0) & i = 3, \ldots, 2l+1, \\ (1, 0, \ldots, 0, 1), & i = 2l+2, \\ (0, \ldots, 0), & i = 0. \end{cases}$$

As usual, a routine computation says that $\mathrm{vol}(\mathcal{P}) = 3/d'!$. Let v be the point

$$\frac{1}{3}\{(v_0, 1) + (v_1, 1) + (v_2, 1)\} + \frac{1}{3}\sum_{q=2}^{\ell+1}(v_{2q}, 1) + \frac{2}{3}\sum_{q=2}^{\ell+1}(v_{2q-1}, 1)$$

belonging to $\mathbb{R}^{d'+1}$. Then

$$v = (1, 1, \ldots, 1, \ell+1) \in \mathbb{Z}^{d'+1}.$$

Thus Lemma 1.1 guarantees that $\delta_{\ell+1} \geq 1$ and $\delta^*_{\ell+1} \geq 1$. Hence $\delta_{\ell+1} = \delta_{\ell+2} = 1$, as required.

(Second Step) Let $k \geq 1$. We write $\mathcal{P} \subset \mathbb{R}^{d'}$ for the integral simplex of dimension d' with the vertices $v_0, v_1, \ldots, v_{3k+2\ell+2}$ as follows:

- $v_0 = (0, 0, \ldots, 0)$,

- $v_1 = (1, 1, 1, 0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, 1, 1, \ldots, 1)$,
- $v_2 = (0, 1, 1, 1, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, 1, 1, \ldots, 1)$,
- $v_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, \underbrace{1}_{(i+1)\text{-th}}, \underbrace{1}_{(i+2)\text{-th}}, 0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$, for $i = 3, 4, 5, \ldots, 3k$,
- $v_{3k+1} = (1, 0, 0, \ldots, 0, 1, \underbrace{1}_{(3k+2)\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$,
- $v_{3k+2} = (1, 1, 0, 0, \ldots, 0, \underbrace{1}_{(3k+2)\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$,
- $v_i = (0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, \ldots, 0, \underbrace{1}_{i\text{-th}}, 0, 1, 0, \ldots, 1, 0)$, for $i = 3k+3, 3k+5, \ldots, 3k+2\ell+1$,
- $v_i = (0, 0, \ldots, \underbrace{0}_{(3k+2)\text{-th}}, \ldots, 0, \underbrace{1}_{i\text{-th}}, 1, 0, 1, 0, \ldots, 1, 0)$, for $i = 3k+4, 3k+6, \ldots, 3k+2\ell+2$.

Let A denote the $(3k+2) \times (3k+2)$ matrix

$$\mathrm{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & \ddots & & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & & \ddots & 1 & 1 & 1 \\ 1 & & & & & 0 & 1 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Then a simple computation on determinants enables us to show that

$$d!\,\mathrm{vol}(\mathcal{P}) = \underbrace{\begin{vmatrix} \mathrm{A} & & * & \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{vmatrix}}_{(3k+2+2\ell)\times(3k+2+2\ell)} = |\mathrm{A}| = 3.$$

One has

$$\begin{aligned} &\frac{1}{3}\{(v_0, 1) + (v_1, 1) + \cdots + (v_{3k+4}, 1)\} + \frac{2}{3}\{(v_{3k+5}, 1) + (v_{3k+7}, 1) + \cdots + (v_{3k+2\ell+1}, 1)\} \\ &\quad + \frac{1}{3}\{(v_{3k+6}, 1) + (v_{3k+8}, 1) + \cdots + (v_{3k+2\ell+2}, 1)\} \\ &= (1, \ldots, 1, k+1, 1, k+2, 1, \ldots, k+\ell, 1, k+\ell+1) \in \mathbb{Z}^{d'+1}. \end{aligned}$$

Hence $\delta_{k+\ell+1} = \delta_{2k+\ell+2} = 1$, as required. □

In order to complete a proof of the "If" part of Theorem 0.1 when $\sum_{i=0}^{d} \delta_i = 3$, we must show the existence of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(1, 0, \ldots, 0, \underbrace{1}_{m\text{-th}}, 0, \ldots, 0, \underbrace{1}_{n\text{-th}}, 0, \ldots, 0)$, where $1 < m < n < d$ and $n - m - 1 \leq m - 1 \leq d - n$.

First, Lemma 3.1 says that there exists an integral convex polytope whose δ-vector coincides with

$$(1, 0, \ldots, 0, \underbrace{1}_{(n-m)\text{-th}}, 0, \ldots, 0, \underbrace{1}_{(2n-2m)\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{3n-3m}.$$

Second, Lemma 3.2 guarantees that there exists an integral convex polytope whose δ-vector coincides with

$$(1, 0, \ldots, 0, \underbrace{1}_{m\text{-th}}, 0, \ldots, 0, \underbrace{1}_{n\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{n+m}.$$

Finally, by using Lemma 1.1, there exists an integral convex polytope $\mathcal{P}$ of dimension d with

$$\delta(\mathcal{P}) = (1, 0, \ldots, 0, \underbrace{1}_{m\text{-th}}, 0, \ldots, 0, \underbrace{1}_{n\text{-th}}, 0, \ldots, 0) \in \mathbb{Z}^{d+1},$$

as desired.

References

[1] E. Ehrhart, Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire, Birkhäuser, Boston, Basel, Stuttgart, 1977.
[2] T. Hibi, Algebraic Combinatorics on Convex Polytopes, Carslaw Publications, Glebe NSW, Australia, 1992.
[3] T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, Adv. Math. 105 (1994) 162–165.
[4] P.R. Scott, On convex lattice polygons, Bull. Aust. Math. Soc. 15 (1976) 395–399.
[5] A. Stapledon, Inequalities and Ehrhart δ-vectors, Trans. Amer. Math. Soc. 361 (2009) 5615–5626.
[6] R.P. Stanley, Enumerative Combinatorics, 1, Wadsworth & Brooks, Cole, Monterey, Calif, 1986.
[7] R.P. Stanley, Decompositions of rational convex polytopes, Ann. Discrete Math. 6 (1980) 333–342.
[8] R.P. Stanley, On the Hilbert function of a graded Cohen–Macaulay domain, J. Pure Appl. Algebra 73 (1991) 307–314.